



Chair for Management Sciences  
and Energy Economics

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EWL Working Paper No. 01/2012

**GAS STORAGE VALUATION:  
A COMPARATIVE SIMULATION STUDY**

by

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June 2012

# Gas storage valuation: A comparative simulation study

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## Abstract

The purpose of this paper is the comparative analysis of four natural gas storage valuation approaches. In competitive natural gas markets the optimal valuation and operation of natural gas storages is a key task for natural gas companies operating storages. Within this paper, four spot based valuation approaches are analyzed regarding computational time and accuracy. In particular, explicit and implicit finite differences, multinomial recombining trees, and Least Squares Monte Carlo Simulation are compared. These approaches are applied to the valuation of a gas storage facility considering three different underlying price processes. Major characteristics of historical natural gas prices are: seasonality, mean reversion and jumps. Therefore, we consider a mean reversion process as underlying price process. In a first step, we extend this mean reversion process to a mean reversion jump diffusion process, to account for jumps, occurring in historical gas spot price time series. Moreover, we consider a more general price process accounting for mean reversion as well as seasonal patterns as observed in the historical time series. Besides the analysis of the numerical results, the benefits and drawbacks of the methodologies are discussed.

*Keywords : natural gas valuation, limited liquidity*

*JEL-Classification: C61, L95, Q40*

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*The authors are solely responsible for the contents which do not necessarily represent the opinion of the Chair for Management Sciences and Energy Economics.*

## 1 Introduction

Assuming unchanged energy policies, the International Energy Agency (IEA) expects the European natural gas demand to increase by 0.8% per year on average up to 2030 (IEA (2009)). Major parts of this growth will be caused by new gas-fired power plants (Kjärstad and Johnsson (2007)). The rising natural gas demand, induced by increasing gas-fired power generation, has two effects: At first, the additional demand for power generation may dampen the seasonality of natural gas demand which is so far dominated by temperature dependent demand for residential heating. On the other hand, short start up and shut down times of natural gas power plants may increase the short term volatility of natural gas demand.

To cope with these possible increasing short term demand fluctuations, gas storages with high deliverability, capturing short term demand variations, need to be developed. Furthermore, facing an increasing total demand for natural gas, additional natural gas storage capacity has to be installed, to keep the current degree of security of supply (cf. Höffler and Kübler (2007)).

In addition to growing demand and decreasing European natural gas supply, European natural gas markets are undergoing liberalization. Consequently, investors are confronted with competitive markets and the valuation and operation of new and existing natural gas storage facilities has to take market information into account. In the past, the operation of a natural gas storage was solely based on the demand fluctuations. In a competitive market, the storage operator must however consider the market information aggregated in the market prices. Facing these profound changes in the natural gas market, the correct valuation of existing storage facilities and investment opportunities in new storage facilities is a major challenge for natural gas utilities.

In a natural gas market with uncertain gas prices, storages offer the possibility to react flexibly on changing prices. An appropriate concept for valuing this flexibility may be the Real Option approach (cf. Dixit and Pindyck (1994)). In addition to the discounted cash flow valuation this approach takes the options offered by an asset into account for the valuation.

In particular, a natural gas storage facility offers the storage operator the option to inject gas when prices are low and to release this gas when prices are high enough. Consequently, adequate valuation approaches are based on market information and thus on market prices to account for the value of flexibility. These valuation approaches may then be classified into intrinsic and extrinsic valuation methodologies.

The intrinsic valuation realizes the seasonal spread of gas prices, buying convenient futures (cf. De Jong and Walet (2005), Breslin et al. (2008)). As futures prices for summer months are below

the prices for winter months, the intrinsic strategy goes long in the cheapest summer month and short in the winter month with the highest futures prices. This approach presets the strategy for the total valuation period at the beginning of the valuation period. The resulting intrinsic value is certain and does not account for possible future price developments. Therefore, the value derived employing this methodology does not include the value of flexibility.

To account for changes in the forward curve, the intrinsic strategy may be extended to the rolling intrinsic strategy. Similar to the intrinsic strategy this strategy realizes the best summer winter spread at the beginning of the valuation period. However, if the forward curve changes over time, existing positions may be rebalanced, resulting in new contracted positions. Thereby the initial hedge is solely adjusted if the additional value resulting from rebalancing exceeds the costs for rebalancing (cf. [Breslin et al. \(2008\)](#)). Hence, the rolling intrinsic value cannot fall below the static intrinsic value (cf. [Gray and Khandelwal \(2004\)](#)).

The static intrinsic hedge is based on the market data available at the initial point. By contrast, the valuation of a rolling intrinsic hedge applies a simulated forward curve, returning an expected value. [Bjerk Sund et al. \(2011\)](#) value a storage facility applying a six factor model using the rolling intrinsic approach. However, even the rolling intrinsic valuation just reacts on changing market conditions and does not anticipate them. This is one major deficiency compared to the extrinsic valuation (cf. [Gray and Khandelwal \(2004\)](#)).

In addition to the intrinsic valuation, the extrinsic valuation also quantifies the value of flexibility offered by a storage facility. The extrinsic valuation maximizes the storage value at a given time step with respect to all possible future spot price developments. [Geman \(2007\)](#) gives a brief introduction to this optimization problem. At each time step the storage operator decides whether it is favorable to buy and store gas, to do nothing and remain at the current storage status or to sell gas at the spot market and to withdraw this gas. This operating decision affects the storage inventory of all subsequent time steps. Thus, the storage operator must take the expected continuation value as well as the immediate cash flow into account of his operating decision. The resulting path dependent optimization problem may be formulated as dynamic optimization problem.

[Byers \(2006\)](#) and [Holland \(2007\)](#) provide a Monte Carlo framework for the valuation of storages. However, Monte-Carlo Simulation implies perfect foresight for each singular price path. Thus, Monte Carlo Simulation is not suited for path dependent options like American options (cf. [Hull \(2009\)](#)), or storages. It will in general overestimate the storage value.

The storage valuation problem may be solved by deriving a partial differential equation and solving this equation analytically, cf. [Hodges \(2004\)](#), or numerically, for instance by finite differences (cf. [Thompson et al. \(2009\)](#), [Chen and Forsyth \(2010\)](#)). [Schlüter and Davison \(2010\)](#) extend the approach of [Thompson et al. \(2009\)](#) and apply a two factor model as underlying price process. In particular, they apply a mean reverting process with a time varying volatility as underlying price process to the optimization problem.

Alternatively one may solve the optimization problem by direct application of the dynamic programming property of the optimization problem. [Weston \(2002\)](#) solves the dynamic programming problem via stochastic dynamic programming on a regular grid. [Maragos \(2002\)](#) applies dynamic programming to a simulated forward curve. [De Jong and Walet \(2005\)](#), [Boogert and de Jong \(2008\)](#), and [Ludkovski and Carmona \(2010\)](#) propose a Least Squares Monte Carlo approach, which was first applied to financial options by [Longstaff and Schwartz \(2001\)](#). Thereby the continuation value of the dynamic programming problem is estimated via a least squares regression. Alternatively, the continuation value can be approximated using price trees (e.g. [Manoliu \(2004\)](#), [Secomandi \(2010\)](#) and [Felix and Weber \(2012\)](#)). In particular, [Felix and Weber \(2012\)](#) aggregate a set of Monte Carlo simulations to a multinomial recombining tree and apply stochastic dynamic programming via recombining trees to the valuation problem.

The aim of this work is to compare the partial differential equation based approaches with the approaches which implement the dynamic programming property directly. Thereby we consider valuation results, computation speed and discretization sensitivity applying different price processes as underlying.<sup>1</sup> In detail, we study explicit and implicit finite differences, Least Squares Monte Carlo Simulation and recombining trees.

The remainder of this article is organized as follows: Section 2 first states the mathematical formulation of the storage optimization problem. Based on this theoretical model, the valuation and approximation approaches are introduced. Section 3 analyzes the different valuation approaches for an exemplary gas storage and three different price processes. Finally, the last section summarizes the results of the paper.

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<sup>1</sup>For a comparison of valuation techniques for financial options see [Geske and Shastri \(1985\)](#).

## 2 Storage Valuation

This section first states the mathematical formulation of the storage optimization problem. Afterwards the analyzed storage valuation approaches are introduced with respect to the mathematical problem formulation.

### 2.1 Mathematical model formulation

For sake of generality, we assume a mean reversion jump diffusion process as underlying spot price process. This process allows to account for two major characteristics observed in historical spot price data: rare jumps and mean reversion to an equilibrium price:<sup>2</sup>

$$\frac{dS}{S} = \kappa(\theta - \lambda\eta - \ln S)dt + \sigma dW + (\phi - 1)dq. \quad (1)$$

Here  $S$  denotes the spot price,  $\kappa$  the mean reversion rate,  $\theta$  the mean reversion level,  $\sigma$  the volatility and  $dW$  the increment of a Brownian motion. The last term describes the jump-part of the process. The jump size  $\phi$  is assumed to be log-normal distributed with  $\ln(\phi) \sim N(\mu_s, \sigma_s^2)$ . The increment of a Poisson process  $dq$  is defined as follows:

$$dq = \begin{cases} 1 & \text{with probability } \lambda dt, \\ 0 & \text{with probability } (1 - \lambda)dt. \end{cases} \quad (2)$$

Here  $\lambda$  is the relative frequency of the jumps occurring in one time step and  $\eta$  denotes the expected percentage price change after a price jump,  $\mathbf{E}[\phi - 1]$ .<sup>3</sup>

Given the underlying price process, the storage operator optimizes the sum of all revenues, resulting from withdrawal, and costs, caused by injection, over the total valuation period. We formulate the storage optimization problem in line with [Thompson et al. \(2009\)](#):

$$V(t_0, S, v(t_0)) = \max_c \mathbf{E} \left[ \int_{t_0}^T (c - a(c)) S e^{-\rho(\tau - t_0)} d\tau \right]. \quad (4)$$

<sup>2</sup>The focus of this paper is the investigation of possible storage *valuation* approaches, hence we limit our analysis to one factor spot price models.

<sup>3</sup>Defining  $x = \ln S$ , equation (1) can be transformed to the following form (cf. [Schwartz \(1997\)](#), [Cartea and Figueroa \(2005\)](#)):

$$dx = \tilde{\kappa}(\tilde{\theta} - \lambda\eta - x)dt + \sigma dW + \ln(\phi)dq, \quad (3)$$

with  $\tilde{\kappa} = \kappa$ ,  $\tilde{\theta} = \theta - \frac{1}{2\kappa}\sigma^2$  and  $\ln(\phi)$  normally distributed. Thus, one may also solve the valuation problem in log-transformed formulation (cf. [Brennan and Schwartz \(1978\)](#), [Geske and Shastri \(1985\)](#)).

Here  $a$  denotes the loss rate dependent on the storage control  $c$ , the latter being negative for injection and positive for withdrawal. Hence, the inventory dynamics are given by

$$dv = -cdt. \quad (5)$$

For the valuation of a physical storage, the technical constraints of this storage have to be taken into account. Consequently, admissible storage controls must satisfy the capacity constraints  $c_{\min}$  and  $c_{\max}$ .<sup>4</sup>

$$c_{\min} \leq c \leq c_{\max}. \quad (6)$$

Substituting  $t_0$  in equation (4) by  $t$  and splitting the interval  $[t, T]$  into  $[t, t + dt]$  and  $[t + dt, T]$  the dynamic programming property of the storage optimization problem can be stated:

$$V(t, S, v) = \max_c E \left[ \int_t^{t+dt} (c - a(c)) S e^{-\rho(\tau-t)} d\tau + V(t + dt, S + dS, v + dv) \right]. \quad (7)$$

Accordingly, the optimal value at time step  $t$  is the sum of the cash flow within the small time interval  $[t, t + dt]$  (resulting from executing the strategy  $c$  at this time step) and the expected optimal value generated in the subsequent time steps after executing the optimal strategy at time step  $t$  (continuation value). Equation (7) illustrates the path dependency of the storage optimization problem. The operating decision at time step  $t$  directly affects the storage inventory and thus the operating decision of all subsequent time steps. Therefore, all time steps are interlinked by the storage inventory  $v$ .

Following [Thompson et al. \(2009\)](#) equation (7) can be reformulated as partial differential equation:

$$0 = \max_c \left[ \frac{1}{2} \sigma^2 S^2 V_{SS} + V_t + \kappa (\theta - \lambda \eta - \ln S) S V_S - c V_v \right. \\ \left. + (c - a(c)) S - \rho V + \lambda V^* \right], \quad (8)$$

with  $V_{SS}$  denoting the second derivative of the storage value with respect to  $S$ . Analogously  $V_S$ ,  $V_t$  and  $V_v$  represent the first derivative of the storage value with respect to  $S$ ,  $t$  and  $v$ , respectively.  $V^*$  represents the expected incremental value after a price jump (cf. [Briani et al. \(2004\)](#)):

$$V^*(t, S, v) = \int_0^\infty V(t, S\phi, v) \tilde{\Gamma}_{\sigma_s}(\phi) d\Phi - V, \quad (9)$$

<sup>4</sup>The storage inventory level may affect the pressure in the underground. Hence, the capacity constraints may depend on the storage inventory. However, for ease of notation we suppress this dependency in the following.

whereas  $\tilde{\Gamma}_{\sigma_s}(\phi)$  is the probability density of the log-normally distributed jump size:

$$\tilde{\Gamma}_{\sigma_s}(\phi) = \frac{1}{\sqrt{2\pi}\sigma_s\phi} e^{-\frac{1}{2}\frac{(\ln\phi - \mu_s)^2}{\sigma_s^2}}. \quad (10)$$

Defining the loss rate  $a(c)$  by

$$a(c) = \begin{cases} l_w c & \text{for } c > 0, l_w > 0, \\ -l_i c & \text{for } c < 0, l_i > 0, \end{cases} \quad (11)$$

and following the arguments in [Thompson et al. \(2009\)](#) the optimal storage strategy results as

$$c_{opt} = \begin{cases} c_{\min} & \text{for } S(1 + l_i) < V_v, \\ c_{\max} & \text{for } S(1 - l_w) > V_v, \\ 0 & \text{else.} \end{cases} \quad (12)$$

According to equation (12) the storage always operates at the maximum capacity constraints. This, so called 'bang-bang', strategy is optimal as the costs and revenues in equation (8) are linear with respect to the optimal control, (cf. [Ludkovski and Carmona \(2010\)](#)).

Moreover, equation (12) shows the dependency of the operating decision on the partial derivative  $V_v$ . This derivative may be seen as opportunity cost for the storage operator. For instance, if the partial derivative exceeds the gas price plus the operating costs, the optimal strategy is injection. In this case the additional value  $V_v$  resulting from an injection exceeds the total injection costs.

To obtain a unique solution for  $V$  (as described by equation (8)) it is necessary to impose the following boundary conditions (cf. [Thompson et al. \(2009\)](#)):

$$(c + a(c)) \geq 0 \quad \text{for } v(t) = C, \quad (13)$$

$$(c + a(c)) \leq 0 \quad \text{for } v(t) = 0, \quad (14)$$

$$V_{SS} \rightarrow 0 \quad \text{for } S \text{ large}, \quad (15)$$

$$V_{SS} \rightarrow 0 \quad \text{for } S \rightarrow 0, \quad (16)$$

$$V(T, S, v(T)) = 0. \quad (17)$$

Equations (13) and (14) specify that the storage can neither be filled above the maximum capacity  $C$  nor emptied below the minimum capacity zero. Conditions (15) and (16) imply that the storage

strategy is invariant to a small price change for very small/large prices (cf. [Thompson et al. \(2009\)](#), [Schlüter and Davison \(2010\)](#)). Equation (17) represents the assumption of a scrap value equal to zero at the end of the valuation period. Subject to these boundary conditions, equation (8) cannot be solved analytically. Therefore the storage value has to be approximated.

Prior to presenting the results of the application in the following a brief introduction to the analyzed approximation and valuation approaches is given. We start with the approximation by finite differences, go along with the valuation by recombining trees and conclude with the explanation of the Least Squares Monte Carlo Simulation.

## 2.2 Finite Differences

Partial differential equations may be approximated by a variety of numerical approximation schemes. However, the most straight-forward one in terms of implementation is the explicit finite differencing scheme. This scheme is therefore also preferred by practitioners and with thus be considered here. In addition, the implicit finite differencing scheme is evaluated. Alternatively one may apply the Crank-Nicolson method. This approach, combining the explicit and implicit schemes, is unconditionally stable. Furthermore, the Crank-Nicolson method is convergent and consistent of order two, whereas the explicit and implicit approach are consistent and convergent of order one. However, as the Crank-Nicolson method can cause spurious oscillations in the approximated solution and as it is complex to implement, we refrain from considering this method.

### 2.2.1 Explicit Finite Differences

Explicit finite differences approximate the partial derivatives of the partial differential equation (8) on the interior points of a storage content, price and time grid. We define these grid points as follows (cf. [Munz and Westermann \(2006\)](#)):

$$\begin{aligned}
 (t_k, S_j, v_i) \text{ with } \quad & t_k = (k-1) \Delta t, \quad k = 1, 2, \dots, K \text{ and } \Delta t = \frac{T}{K-1} \\
 & v_i = (i-1) \Delta v, \quad i = 1, 2, \dots, I \text{ and } \Delta v = \frac{C}{I-1} \\
 & S_j = S_{\min} + (j-1) \Delta S, \quad j = 1, 2, \dots, J \text{ and } \Delta S = \frac{S_{\max} - S_{\min}}{J-1}.
 \end{aligned} \tag{18}$$

Thereby time horizon and storage volume are discretized into  $K$  and  $I$  steps, respectively. Moreover,  $S_{\max}$  ( $S_{\min}$ ) denotes the maximum (minimum) price grid point in the price direction taken into account for the valuation and  $J$  is the number of price grid points. In the following we indicate the

value of the storage for a grid point  $(t_k, S_j, v_i)$  by  $V_{i,j}^k$ . Hence, equation (8) may be rewritten using explicit finite differences:

$$\begin{aligned}
-\frac{V_{i,j}^{k+1} - V_{i,j}^k}{\Delta t} &= \frac{1}{2} \sigma^2 S_j^2 \frac{V_{i,j+1}^{k+1} - 2V_{i,j}^{k+1} + V_{i,j-1}^{k+1}}{\Delta S^2} \\
&+ \kappa(\theta - \lambda \eta - \ln S_j) S_j \frac{V_{i,j+1}^{k+1} - V_{i,j-1}^{k+1}}{2\Delta S} \\
&- c_{i,j}^k \frac{V_{i+1,j}^{k+1} - V_{i-1,j}^{k+1}}{2\Delta v} + \left( c_{i,j}^k - a(c_{i,j}^k) \right) S_j \\
&- \lambda V_{i,j}^{k+1} - \rho V_{i,j}^{k+1} + \lambda \mathbf{E}^\phi \left[ \sum_{m=1}^J V_{i,m}^{k+1} \right].
\end{aligned} \tag{19}$$

The last term in equation (19) accounts for the expected value after a jump has occurred and  $J$  denotes the number of grid points in the price directory (cf. definition (18)). In equation (19), forward differences are used for  $V_i$  and central differences are used for  $V_v$ ,  $V_{SS}$  and  $V_S$ . The last term, representing an integral, is calculated applying the trapezoid rule (cf. Thompson et al. (2009)). Starting at the last time step  $T$  and proceeding backwards, the value at time step  $k$  is calculated utilizing only the known grid points at time step  $k+1$ .

The explicit finite differencing scheme is easy to implement. Though it has the drawback of being only conditional stable. To derive the stability conditions we rewrite equation (19) as follows:

$$\begin{aligned}
V_{i,j}^k &= \Delta t \left( V_{i,j+1}^{k+1} \left( \frac{1}{2} \frac{\sigma^2 S_j^2}{\Delta S^2} + \frac{\kappa(\theta - \lambda \eta - \ln S_j) S_j}{2\Delta S} \right) \right. \\
&+ V_{i,j}^{k+1} \left( \frac{1}{\Delta t} - \frac{\sigma^2 S_j^2}{\Delta S^2} - \lambda - \rho \right) \\
&+ V_{i,j-1}^{k+1} \left( \frac{1}{2} \frac{\sigma^2 S_j^2}{\Delta S^2} - \frac{\kappa(\theta - \lambda \eta - \ln S_j) S_j}{2\Delta S} \right) \\
&- c_{i,j}^k \frac{V_{i+1,j}^{k+1}}{2\Delta v} + c_{i,j}^k \frac{V_{i-1,j}^{k+1}}{2\Delta v} + \lambda \mathbf{E}^\phi \left[ \sum_{m=1}^J V_{i,m}^{k+1} \right] \\
&\left. + \left( c_{i,j}^k - a(c_{i,j}^k) \right) S_j \right).
\end{aligned} \tag{20}$$

To derive the stability condition it is sufficient to focus on the first three terms of equation (20) (cf. Ames (1992) and Briani et al. (2004)). According to Ames (1992) the coefficients of  $V_{i,j+1}^{k+1}$ ,  $V_{i,j}^{k+1}$  and  $V_{i,j-1}^{k+1}$  in equation (20) must be positive. Thus we obtain the following stability condition:<sup>5</sup>

$$\sqrt{\frac{\Delta t \sigma^2 S_{\max}^2}{1 - (\lambda + \rho) \Delta t}} \leq \Delta S \leq \min_{j=1, \dots, J} \frac{\sigma^2 S_j}{\{\kappa |\theta - \lambda \eta - \ln S_j|\}}. \tag{21}$$

<sup>5</sup>These conditions are derived in the Appendix.

This condition depends on the price  $S_j$ . The upper bound of  $\Delta S$  depends on the absolute difference of  $(\theta - \lambda \eta)$  and the price  $S_j$ . To obtain the binding upper bound,  $S_j$  must be chosen to minimize the right hand side of the inequality chain.

Equation (21) shows the interdependency of time and price grid step size. Hence, the price step size has to be selected with respect to the time step size. This may lead to particularly small time steps to obtain an acceptable degree of accuracy.

## 2.2.2 Implicit Finite Differences

In contrast to explicit finite differences implicit finite differencing schemes are unconditional stable. Thus the time, price, and inventory discretization step size can be chosen independently.

Implicit finite differences approximate the differential equation at time step grid point  $k$  using the grid points of the same time step  $k$ :

$$-\frac{V_{i,j}^{k+1} - V_{i,j}^k}{\Delta t} = \frac{1}{2} \sigma^2 S_j^2 \frac{V_{i,j+1}^k - 2V_{i,j}^k + V_{i,j-1}^k}{\Delta S^2} + \kappa(\theta - \lambda \eta - \ln S_j) S_j \frac{V_{i,j+1}^k - V_{i,j-1}^k}{2\Delta S} \quad (22)$$

$$- c_{i,j}^k \frac{V_{i+1,j}^k - V_{i-1,j}^k}{2\Delta v} + \left( c_{i,j}^k - a(c_{i,j}^k) \right) S_j - \lambda V_{i,j}^k - \rho V_{i,j}^k + \lambda \mathbf{E}^\phi \left[ \sum_{m=1}^J V_{i,m}^k \right]. \quad (23)$$

Reformulating equation (23) we get

$$\begin{aligned} V_{i,j}^{k+1} &= -\Delta t \left( V_{i,j+1}^k \left( \frac{1}{2} \frac{\sigma^2 S_j^2}{\Delta S^2} + \frac{\kappa(\theta - \lambda \eta - \ln S_j)}{2\Delta S} \right) \right. \\ &+ V_{i,j}^k \left( -\frac{1}{\Delta t} - \frac{\sigma^2 S_j^2}{\Delta S^2} - \lambda - \rho \right) \\ &+ V_{i,j-1}^k \left( \frac{1}{2} \frac{\sigma^2 S_j^2}{\Delta S^2} - \frac{\kappa(\theta - \lambda \eta - \ln S_j) S_j}{2\Delta S_j} \right) \\ &- c_{i,j}^k \frac{V_{i+1,j}^k}{2\Delta v} + c_{i,j}^k \frac{V_{i-1,j}^k}{2\Delta v} + \lambda \mathbf{E}^\phi \left[ \sum_{m=j}^J V_{i,m}^k \right] \\ &\left. + \left( c_{i,j}^k - a(c_{i,j}^k) \right) S_j \right) \end{aligned} \quad (24)$$

The determination of the storage value  $V_{i,j}^k$  requires the solution of a system of equations at each time step, as the interior grid points at the current time step  $k$  are unknown. The resulting matrix, characterizing this system of equations, has the size  $(IJ) \times (IJ)$  (cf. equation (24)). This system of equations may be solved applying a LR-factorization to this matrix. If the matrix does not change

over time, this factorization needs to be done only once - saving computational time and space. However, for the storage valuation some matrix entries depend on the optimal storage control  $c_{i,j}^k$  and therefore may change over time. Therefore, the system of equations has to be solved for each time step separately.

The solution of a linear system of equations at each considered time step leads to a high computation time. This computation time depends to a large part on the matrix size which grows quadratically in the number of price and inventory grid steps (see above). However, as implicit finite differences are unconditionally stable, the computation time may be reduced by decreasing the number of time steps.

Finite differences approximate the partial differential equation (8) on a time, inventory and price grid. Thereby all grid points are equidistant. The recombining tree approach approximates the optimal storage value also on a time, inventory and price grid. However, the recombining tree (which represents the price grid) is constructed applying a clustering algorithm to *simulated prices*. Therefore, the resulting price grid for the recombining tree approach is not necessarily equidistant. Moreover, recombining trees and Least Squares Monte Carlo Simulation approximate the dynamic programming equation (7) whereas the finite differencing schemes focus on the approximation of the partial differential equation (8).

### 2.3 Recombining Trees

Trees are a standard approach to value path dependent American options in finance. In contrast to the previously introduced numerical approximation schemes, trees do not approximate the partial differential equation (8) but implement the dynamic programming principle described in (7). In the following we give a brief introduction to the valuation by numerically constructed recombining trees based on [Felix and Weber \(2012\)](#).

The multinomial recombining trees applied in the valuation procedure are built up on a set of Monte Carlo simulations. These Monte Carlo simulations are aggregated to a predefined number  $\bar{J}$  of price clusters  $s_t$  at each time step. In addition, the probabilities for each price cluster at each time step  $P(s_t)$  and the transition probabilities  $P_{Tr}(s_t, s_{t+1})$  in between the price clusters of adjacent time steps are determined. These probabilities and price clusters build the multinomial recombining price tree (cf. [Weber \(2005\)](#)). As it is based on Monte-Carlo simulations this approach allows to account for a large variety of price processes requiring solely the Markov property for the price process.

For a given recombining tree, the storage value at time step  $t$ , price cluster  $s_t(j)$ , ( $j \in \{1, \dots, \bar{J}\}$ ),

and inventory level  $v_t(i)$  ( $i \in \{1, \dots, I\}$ , cf. (18)) is calculated as sum of the instant cash flow  $V_d(t, s_t(j), v_t(i))$  and the expected continuation value resulting from the operating decision at time step  $t$  (cf. equation (7)):

$$\begin{aligned} \bar{V}(t, s_t(j), v_t(i)) &= \max_c \{V_d(t, s_t(j), v_t(i)) \\ &+ \sum_{j'=1}^J P_{tr}(s_t(j), s_{t+1}(j')) \bar{V}(t+1, s_{t+1}(j'), v_{t+1}(i'))\}, \\ &\forall t \in \{t_0, \dots, T-1\}, s_t(i) \in \bar{S}_t, v_t(i) \in \{1, \dots, C\}. \end{aligned} \quad (25)$$

Here  $v_{t+1}(i')$  is the storage inventory resulting from executing the optimal operating decision at time step  $t$  and  $\bar{S}_t$  denotes the set of all price clusters at time step  $t$ . The immediate cash flow in (25) results directly from the operating decision  $c$  at time step  $t$ :

$$V_d(t, s_t(i), v(t)) = (c(t, s_t(j), v_t(i)) - a(c(t, s_t(j), v_t(i)))) s_t(j) \quad (26)$$

Applying the loss rates defined in (11) the immediate cash flow is rewritten as follows:

$$V_d(t, s_t(i), v(t)) = \begin{cases} c(t, s_t(j), v_t(i)) (1 + l_i) s_t(j) & \text{for } c < 0, \\ c(t, s_t(j), v_t(i)) (1 - l_w) s_t(j) & \text{for } c > 0, \\ 0 & \text{for } c = 0. \end{cases} \quad (27)$$

To solve the dynamic optimization problem stated in equation (25), the methodology works backwards in time like the finite differencing approaches. The algorithm starts at the last time step  $T$  with a given scrap value (cf. condition (17)). Then, at each time step the optimal storage strategy and the optimal storage value are determined for all storage inventory levels and price clusters. The optimal storage strategy is determined analogously to equation (12), whereas  $V_v$  is approximated by the expected incremental (decremental) value for injection (withdrawal). The storage value then results as expected storage value at time step zero.

Employing Monte Carlo simulations, this approach can be easily adapted to all price processes satisfying the Markov property. Moreover, due to the recombining structure of the tree, the computational time and space requirements of the valuation procedure are expected to be rather low.

## 2.4 Least Squares Monte Carlo Simulation

This section gives an introduction to the Least Squares Monte Carlo Simulation (LSMC) for natural gas storage valuation as proposed by [Boogert and de Jong \(2008\)](#). In contrast to a Monte Carlo Simulation the LSMC approach does not postulate perfect foresight for each specific price path. Like the recombining tree approach, the Least Squares Monte Carlo Simulation is based on set of  $N$  simulated price paths  $p_t(j), j = \{1, \dots, N\}$  and applies the dynamic programming property of the valuation problem. In fact the LSMC estimates the continuation value in equation (7) applying a least squares regression. Thereby, the continuation values of all price paths are regressed on a set of basis functions  $\Psi_l, l = \{1, \dots, L\}$  of the simulated prices for each storage inventory level. The regression results  $\hat{\alpha}_l, l = \{1, \dots, L\}$  are then used to estimate the continuation value for each price path:

$$\hat{V}(t+1, p_t(j), v_t(i)) = \sum_{l=0}^L \hat{\alpha}_l \Psi_l(p_t(j)) \quad (28)$$

Thereby  $\hat{V}(t+1, p_t(j), v_t(i))$  represents the estimated storage value at time step  $t+1$  with an inventory level  $v_t(i)$  given the price  $p_t(j)$  at time step  $t$ . Starting at the last time step  $T$  and working backwards, the estimated continuation values are applied to determine the optimal operation strategies. Therefore the values of all possible storage operations (withdrawing, injection and 'doing-nothing') are calculated. For instance, the value for withdrawing from the storage at time  $t$ , price path  $p_t(j)$  and storage inventory  $v_t(i)$ ,  $V_{c_{\max}}(t, p_t(j), v_t(i))$  is computed as follows:

$$V_{c_{\max}}(t, p_t(j), v_t(i)) = -p_t(j)(1-l_w) + \hat{V}(t+1, p_t(j), v_t(i-1)). \quad (29)$$

The values of the strategies injection  $V_{c_{\min}}$  and 'doing-nothing'  $V_0$  are determined analogously. The algorithm then selects the strategy which corresponds to the highest value:

$$c_{opt} = \arg \max_{c \in \{c_{\min}, 0, c_{\max}\}} V_c \quad (30)$$

This strategy determines the continuation value of the previous period  $t-1$ . Working backwards in time, the storage value results as average at time step zero over all price paths.

## 3 Application

This section first describes the applied price models as well as the parameter calibration. Afterwards the general model settings are discussed before presenting the comparison results.

### 3.1 Applied Simulation Models

#### 3.1.1 Mean Reversion Jump Diffusion

To account for potential jumps as well as for the mean reversion occurring in the historical time series, we consider a mean reversion jump diffusion process. The parameter estimation is done applying the log-price formulation (cf. equation (3)):

$$d \ln S = \tilde{\kappa} (\tilde{\theta} - \lambda \eta - \ln S) dt + \sigma dW + \ln(\phi) dq. \quad (31)$$

The parameter estimation is done in two steps: In a first step, the jumps are filtered out of the historical time series data and the jump process parameters are estimated. In a second step the parameters of the mean reversion process are calculated based on the filtered data.

The jumps are filtered out of the historical data applying the filtering procedure described in [Weron \(2006\)](#): After defining a threshold, for instance two or three times of the volatility, all differences  $(\ln S_t - \ln S_{t-1})$  exceeding this threshold are sorted out. Thereby, the jump frequency, the mean jump size, and the volatility of the jumps size are obtained. In addition, the volatility of the filtered set, without jumps, can be calculated. Applying this volatility, the filtering algorithm starts again. This procedure repeats until the jump frequency does not change within a given precision.

According to [Dixit and Pindyck \(1994\)](#) the non-jump part of equation (31) may be seen as continuous version of an AR(1) process. Thus, the process parameters are estimated running the following regression on the filtered time series data:

$$\ln S_t - \ln S_{t-1} = a + b \ln S_{t-1} + \varepsilon_t. \quad (32)$$

Applying the estimation results, the parameters of the mean reversion part are calculated as follows:<sup>6</sup>  $\tilde{\theta} = -\frac{\hat{a}}{\hat{b}}$ ,  $\tilde{\kappa} = -\ln(1 + \hat{b})$  and  $\sigma = \hat{\sigma}_\varepsilon \sqrt{\frac{2 \ln(1 + \hat{b})}{(1 + \hat{b})^2 - 1}}$ . Afterwards, the parameters of equation (1) are calculated according to the transformation  $\kappa = \tilde{\kappa}$  and  $\theta = \tilde{\theta} + \frac{\sigma^2}{2\kappa}$  (cf. [Schwartz \(1997\)](#)).

The parameters of the Mean Reversion Process without jumps are estimated applying only step two without filtering the historical data.

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<sup>6</sup>The calculation of  $\sigma$  has been adjusted to correct a typo in [Dixit and Pindyck \(1994\)](#).

### 3.1.2 Generalized Mean Reversion Model

Large parts of natural gas demand are still temperature driven and thus exhibit seasonal patterns. Thus, spot prices may also incorporate seasonality as the spot price depends on the demand pattern. We consider this seasonality adjusting a model proposed by [Cartea and Figueroa \(2005\)](#) for the electricity spot market to the natural gas market. Thereby, our simulation model estimates seasonality impacts on natural gas spot prices via a least squares regression of gas prices on a combination of trigonometric functions. The spot pricing simulation model can be separated in the following steps:<sup>7</sup>

1. Transformation of the historical spot price data
2. Determination of deterministic impacts: seasonality
3. Modeling the residuals as Mean Reversion Process with equilibrium level zero.<sup>8</sup>

### 3.2 Case Study

This section applies the valuation methodologies to an exemplary natural gas storage facility. We consider the impact of the price discretization on the storage value as well as on the computational time. Moreover, the impact of different price processes is analyzed. Finally, a forward simulation based on historical spot prices is performed for selected valuation approaches.

We value a storage facility with a working gas capacity of 15 million MWh and identical (inventory independent) withdrawal and injection capacities of 500 thousand MWh per day. This storage offers high flexibility due to high injection and extraction rates. In fact, the storage can be emptied within a month. Accordingly, we apply an inventory discretization of 30 steps.

The proposed price models are calibrated to historical times series data of the Title Transfer Facility (TTF). Thereby, all spot prices in between 2006/03/31 and 2011/03/31 are taken into account. Table 1 shows the calibration results of the price processes - showing that ignoring jumps occurring in the historical price data results in significant higher estimated volatility and mean reversion rate. Moreover considering seasonality in the proposed manner does not decrease the estimated volatility (cf. Table 1) but allows to account for (deterministic) seasonal price movements. The seasonal Mean Reversion level of the general price model is shown in Figure 1. These calibration results are applied to simulate price path covering the valuation horizon of one year.

As we focus on a valuation horizon of one year we do not account for a discount rate. For the

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<sup>7</sup>This model reproduces mean reverting as well as seasonal price patterns. Therefore we refer to this price model as Generalized Mean Reversion model.

<sup>8</sup>The parameter estimation for this Mean Reversion part is done as described above.

Table 1: Parameter calibration: Results for the Mean Reversion (MR), Mean Reversion Jump Diffusion (MR-JD) and generalized price model

	MR	MR-JD	Gen. Price Model
$\kappa$	0.0136	0.0104	0.0140
$\tilde{\theta}$	2.82324	2.8004	0
$\sigma$	0.0582	0.042	0.0582
$\ln(\phi)$	-	$\sim N(0, 0.035)$	-
$\lambda$	-	0.049	-

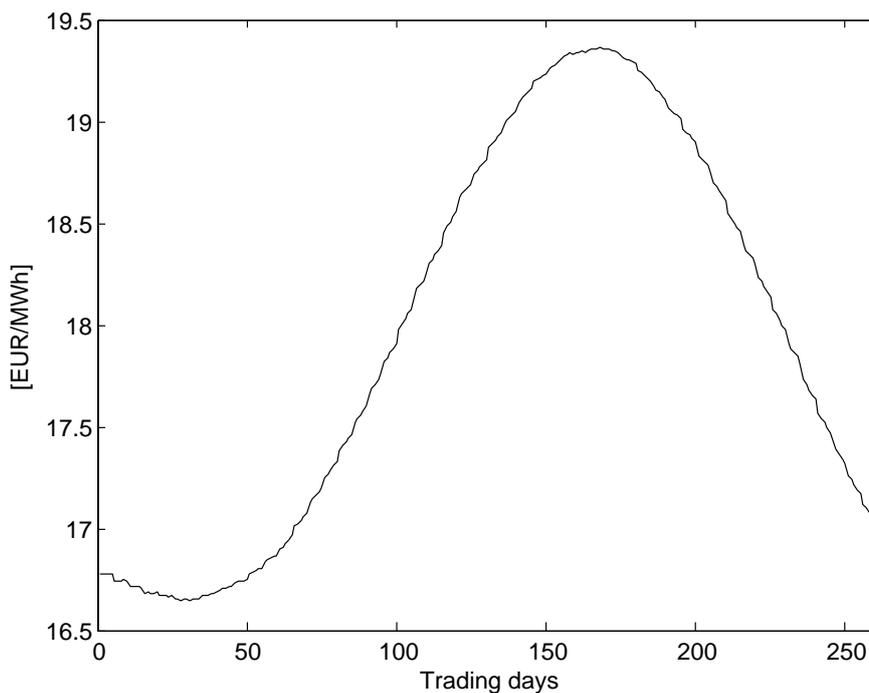


Figure 1: Mean daily prices for the general price model

Least Squares Monte Carlo approach we apply power basis functions up to degree three.<sup>9</sup>

In a first step the value sensitivity with respect to the price discretization is analyzed for the case of a Mean Reversion process. We start with the comparison of the approximation by explicit and implicit finite differences. For a maximum price grid point  $S_{\max} = 40$  and a price discretization  $\Delta S = 1$  we apply a time step size of  $\Delta t = 0.125$  to satisfy the stability condition of the explicit approach (cf. (21)). Thereby, time is measured in days. To guarantee comparability, we apply this time step size as well for the implicit finite differencing approach in the following.

We find that the storage value calculated by both approaches grows with an increase in the number of price grid points (cf. Figure 2). For 39 price grid points (applying  $\Delta S = 1$  and a minimum

<sup>9</sup>Boogert and de Jong (2008) found that applying b-splines or power basis functions yield similar results. Therefore we limit our investigation to power basis functions.

price grid point of 2), the valuation results of implicit and explicit finite differences differ by less than 1%. The results for the explicit finite differences are not shown for more than 47 price discretization steps (corresponding to  $\Delta S = 0.82$ ) as the stability condition (21) is violated for more than 47 price discretization steps.<sup>10</sup> Furthermore the implicit approach obviously converges with a finer price grid. For instance, halving the price step size from  $\Delta S = 0.5$  ( $\Delta S = 0.25$ ) to  $\Delta S = 0.25$  ( $\Delta S = 0.125$ ) leads to storage value increase by 0.7 (0.2)%.<sup>11</sup>

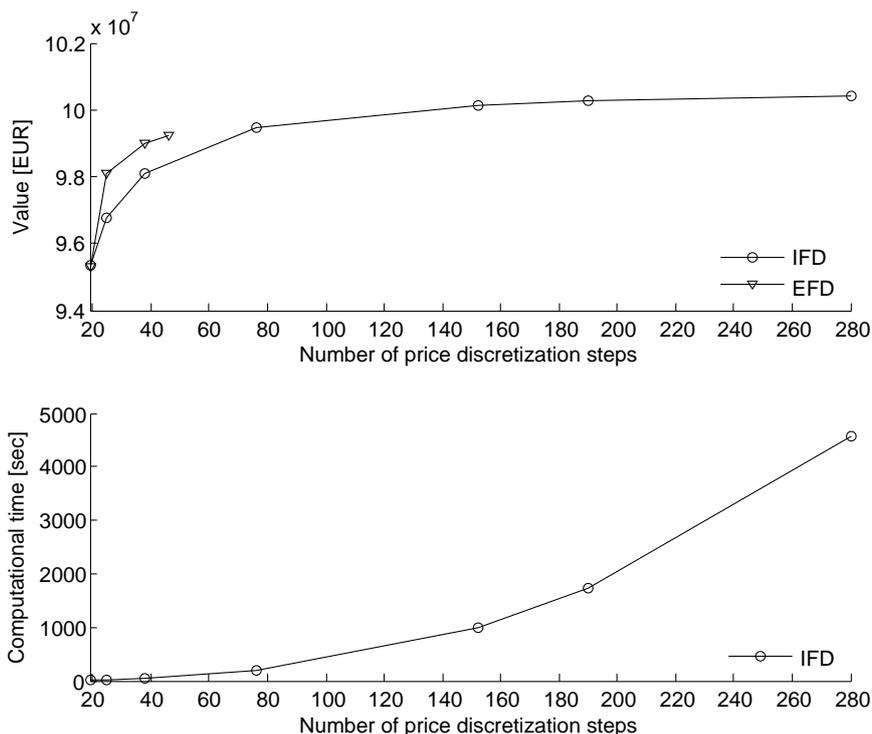


Figure 2: Price step discretization sensitivity for finite differences applying a Mean Reversion process.

The computational time of the explicit approach grows linearly with the number of price grid points. In contrast Figure 2 illustrates a quadratic growth of the computational time for the implicit approach.<sup>12</sup> In particular, the matrix which describes the linear system of equations to be solved for the implicit finite differences, grows quadratically in the number of price steps (cf. section 2.2). Moreover the need for storage space increases as well. For instance, applying a price grid with 200 steps and an inventory discretization of 30 steps the matrix of the system of equations has 36 million entries.<sup>13</sup>

<sup>10</sup>Thereby the expected values at time step  $t_0$  are considered applying the transition density of the Mean Reversion Jump Diffusion as described in Das (2002). Focusing on the option value of the storage, we denote solely the storage values which correspond to an empty storage.

<sup>11</sup>As the maximum price grid point is set to 40 and the minimum price grid point is set to 2 a price grid step size of  $\Delta S = 0.5$  corresponds to 77 price discretization steps.

<sup>12</sup>All valuation approaches are implemented in MATLAB® and operated on a 3.4 GHz Eight-Core desktop PC. As the computational times of the explicit approach are marginal compared to the implicit approach these are not illustrated in Figure 2.

<sup>13</sup>One should note that the implicit finite differences in this application also apply the time step size of  $\Delta t = 0.125$  to guarantee the comparability of the valuation results. As the computational time depends linearly on the time step size  $\Delta t$ , applying a time step size

Besides the comparison of the different finite differencing approaches we compare the valuation by recombining trees and Least Squares Monte Carlo Simulation with the implicit finite difference approach. Thereby we limit our considerations on the implicit finite differencing approach to omit the stability problems of the explicit finite differencing approach for higher price discretizations. We thereby apply an identical time step size of  $\Delta t = 1$ . Figure 3 illustrates the valuation results for the

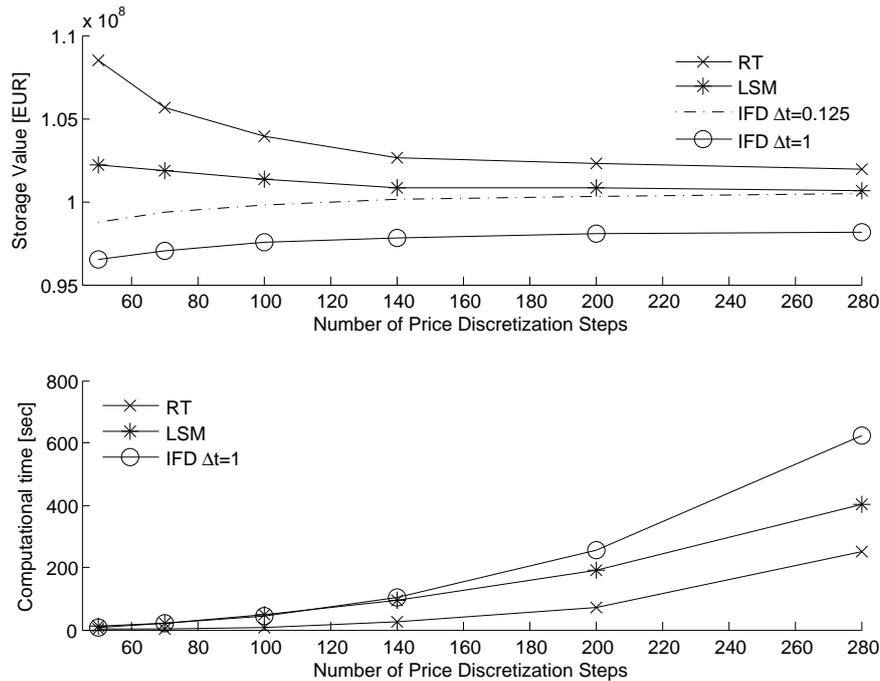


Figure 3: Price step discretization sensitivity for IFD, RT and LSM applying a Mean Reversion process. Thereby  $\bar{K}$  price discretization steps correspond to  $N = 2\bar{K}^2$  price simulations for the Least Squares Monte Carlo Simulation.

three valuation approaches as the number of price discretization steps increases.<sup>14</sup> Thereby the implicit finite differences approximate the storage value from below. In contrast recombining trees as well as Least-Squares-Monte-Carlo simulation converge from above (cf. Figure 3). Applying 50 price discretization steps to the recombining tree approach and the implicit finite differences and 5000 simulations to the Least Squares Monte Carlo simulation the three valuation approaches differ at maximum by 11.1%. Increasing the number of discretization steps to 280 (corresponding to 160000 price simulations) this difference decreases to less than 3.8%. This maximum differ-

of  $\Delta t = 1$  would decrease the computational time of the implicit approach by a factor eight. Accordingly [Munz and Westermann \(2006\)](#) state in general that the explicit approach may be favorable (with respect to computational time and not to accuracy) even if the stability condition leads to a time step discretization which is a tenth of the implicit time step discretization. In our application the explicit approach is favorable (with respect to computational time) for even smaller time steps, as the matrix of the implicit approach has to be solved in each time step separately.

<sup>14</sup>For a given number of price simulations  $N$  we apply the rule of thumb proposed by [Mardia et al. \(1979\)](#) to determine the number of clusters:  $\bar{K} = \sqrt{N/2}$ . We find that clustering the simulations via the percentiles of the simulations performs as well as the k-means algorithm applied in [Felix and Weber \(2012\)](#) but improves the computational time significantly.

ence represents the difference between the recombining tree and the implicit finite differences. The difference between the recombining tree and the Least Squares Monte Carlo Simulation is 1.2%. This difference may be explained by the loss of information resulting from clustering the simulated prices. The remaining difference between the Least Squares Monte Carlo Simulation and the implicit differences is 2.6% for the highest illustrated discretization. This difference may be explained by the discretization error of the finite differencing approach. In particular, as the implicit finite differencing approach approximates the partial differential equation (8), a smaller time step size reduces the local discretization error. Applying a time step size of  $\Delta t = 0.125$  to the implicit finite differences the maximum difference between the three valuation approaches decreases to less than 1.5%.

As depicted in Figure 3 the recombining tree approach requires the lowest computational time of all valuation approaches (including the time for clustering the prices).<sup>15</sup> Reducing the dimensionality by clustering the simulated prices the sole valuation time of the recombining tree approach is less than one second even for the highest discretization level. Thus, regarding computational time aspects, the recombining tree valuation may be even more advantageous if multiple valuation runs (e.g. for different storage facilities) have to be performed with an identical recombining tree. Moreover the recombining tree approach may be beneficial if a high inventory discretization is applied: For instance the valuation time (and the need for computational space) of the implicit finite differences approximation grows quadratically with respect to the number of time and price discretization steps. As the regression is done for each inventory level at each time step separately, computational time and space of the Least Squares Monte Carlo Simulation grow linearly with respect to the inventory discretization. However, applying the recombining tree approach with  $\bar{K}$  nodes determined by  $N$  simulations reduces the need for computational space by the factor  $\bar{K}/N$ . Thus, recombining trees are also favorable for high dimensional problems.

Investigating the impact of the price specification on the storage value (cf. Table 2),<sup>16</sup> we find that for a mean reversion jump diffusion process the storage value decreases by 3.7% on average. We attribute this to the lower estimated volatility of the mean reversion jump diffusion specification (cf. Table 1). Furthermore, the storage value increases by more than 6% on average for the general price model. This increase may be explained as follows: Assuming a seasonally varying mean reversion level adds a deterministic (yearly) price fluctuation to the price simulations but does not alter the daily price volatility. Thus the deterministic fluctuation increases the intrinsic storage

<sup>15</sup>For sake of the distinctness of Figure 3 the computational times of the implicit finite differencing approach are not illustrated. They can be found in Figure 2.

<sup>16</sup>Results for implicit finite differences are shown for  $\Delta t = 0.125$ .

value whereas the extrinsic values remains more or less unchanged - resulting in an increase of the total storage value. The general price model is not applied to implicit finite differences as a seasonally varying mean reversion level is not easily implemented for this approach. Altogether the maximum difference between the storage values of the considered price specifications is 10%.

Table 2: Valuation results for different price models

Price-process	Value [ $10^6$ EUR]		
	IFD	RT	LSMC
MR	100.3	102.3	100.8
MR-JD	96.6	98.9	96.8
Gen. Price Model	-	108.5	107.3

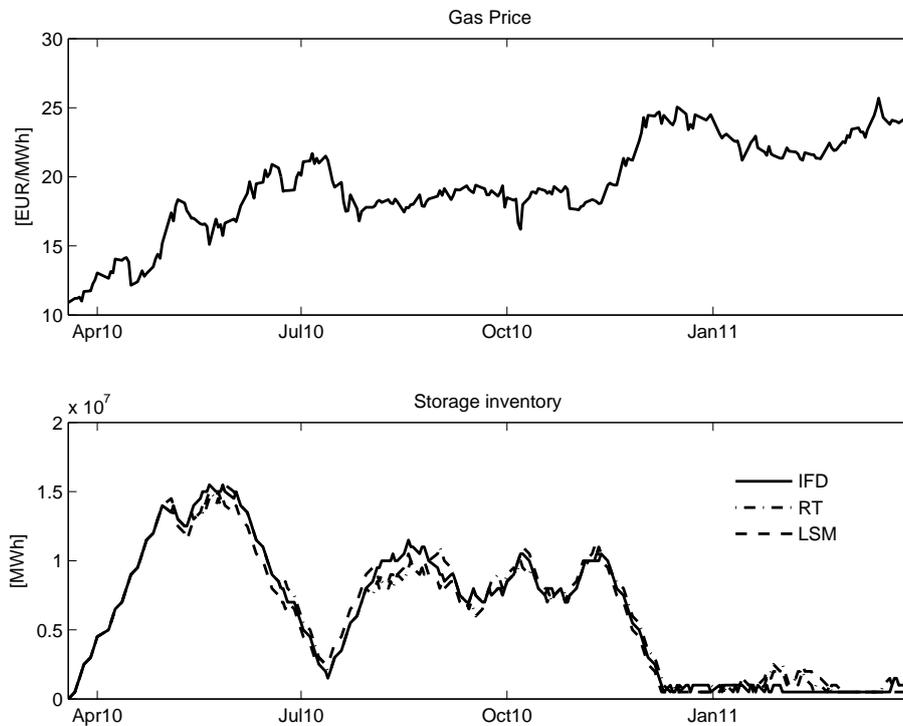


Figure 4: Historical scheduling of the storage.

Figure 4 shows the results of a forward simulation for all valuation methodologies. Thereby, the operating rules, derived from the backward optimization for the Mean Reversion price process, are applied to historical TTF spot price data between 2010/04/01 and 2011/03/31. Starting with an empty storage and the historical spot price at the beginning of the valuation period, the forward simulation determines the optimal operating decision for this price-inventory combination. This optimal operation rule is executed and one obtains the storage inventory of the subsequent time

step. The forward simulation then recursively calculates the optimal storage scheduling applying these two steps to all subsequent time steps.

As Figure 4 illustrates, the forward simulation results are very similar for all valuation approaches. Thereby, the operating decision of the three valuation approaches is determined to a large extent by the continuation value. In addition, the immediate values for injection/withdrawal into/out of the storage are determined by the current (historical) price which is identical for all valuation approaches. Thus, one may deduce from Figure 4 that the continuation values of the three valuation approaches are also similar for all time steps. Moreover, Figure 4 shows that the storage is emptied already at the end of 2010 as the price expectations derived from historical calibration suggest that a further increase of prices has a very small likelihood.

#### 4 Conclusion

In this paper we analyze and compare spot based valuation approaches for natural gas storages. Based on the mathematical optimization formulation we emphasize the relationship of the different valuation approaches. Applying three different price processes to an exemplary gas storage, we find that all approaches lead to similar results. Moreover we investigate the computational performance of the valuation approaches with respect to the price discretization. Thereby, we do not consider the explicit finite differencing approach, as price and time discretization are restricted by stability conditions for this approach. These stability conditions are derived within this paper. We find that in our example the recombining tree approach converges slower than the other approaches. However, the implicit finite differencing scheme as well as the Least Squares Monte Carlo Simulation are very demanding in computational time and storage space. In particular, the computational time of the recombining tree is below the computational time of the other approaches for all price discretizations considered. Since it reduces the dimensionality by clustering the simulations the recombining tree approach is most suited for problems requiring a high inventory discretization and for high dimensional problems (e.g. the valuation of multi-reservoir pumped hydro storage systems with inflow). Finally, a forward simulation is applied to historical time series data, showing similar results for all valuation approaches. Future research may focus on the application of recombining trees to high dimensional problems like the valuation of multi-reservoir pumped hydro storage systems.

#### References

Ames, W. (1992). *Numerical methods for partial differential equations*. (3 ed.). Academic Press, San Diego.

- Bjerk Sund, P., G. Stensland, and F. Vagstad (2011). Gas Storage Valuation: Price Modelling v. Optimization Methods. *The Energy Journal* 32(1), 203–228.
- Boogert, A. and C. de Jong (2008). Gas storage valuation using a Monte Carlo method. *The journal of derivatives* 15(3), 81–98.
- Brennan, M. and E. Schwartz (1978). Finite Difference Methods and Jump Processes Arising in the Pricing of Contingent Claims: A Synthesis. *Journal of Financial and Quantitative Analysis* 13(3), 461–474.
- Breslin, J., L. Clewlow, T. Elbert, C. Kwok, and C. Strickland (2008). Gas Storage: Overview & Static Valuation. *LACIMA Group*.
- Briani, M., C. Chioma, and R. Natalini (2004). Convergence of numerical schemes for viscosity solutions to integro-differential degenerate parabolic problems arising in financial theory. *Numerische Mathematik* 98(4), 607–646.
- Byers, J. W. (2006). Commodity storage valuation: A linear optimization based on traded instruments. *Energy Economics* 28, 275–287.
- Cartea, . and M. G. Figueroa (2005). Pricing in electricity markets: A mean reverting jump diffusion model with seasonality. *Applied Mathematical Finance* 12(4), 313–335.
- Chen, Z. and P. Forsyth (2010). Implications of a regime-switching model on natural gas storage valuation and optimal operation. *QUANTITATIVE FINANCE* 10(2), 159–176. Article 550ZU English Times Cited:0 Cited References Count:39.
- Das, S. (2002). The surprise element: jumps in interest rates. *Journal of Econometrics* 106(1), 27–65.
- De Jong, C. and K. Walet (2005). Gas Storage Management. In V. Kaminski (Ed.), *Managing Energy Price Risk* (3 ed.), pp. 631–648. London: Risk Books.
- Dixit, A. K. and R. S. Pindyck (1994). *Investment under Uncertainty*. Princeton, N.J.: Princeton University Press.
- Felix, B. and C. Weber (2012). Gas storage valuation applying numerically constructed recombining trees. *European Journal of Operational Research* 216(1), 178–187.
- Geman, H. (2007). *Commodities and commodity derivatives: modeling and pricing for agriculturals, metals and energy* (Repr. ed.). Wiley finance series. Chichester: Wiley & Sons.
- Geske, R. and K. Shastri (1985). Valuation by Approximation: A Comparison of Alternative Option Valuation Techniques. *Journal of Financial and Quantitative Analysis* 20(1), 45–71.
- Gray, J. and P. Khandelwal (2004, September). Realistic natural gas storage models II: Trading strategies. *Commodities Now*, 1–5.
- Höfler, F. and M. Kübler (2007). Demand for storage of natural gas in northwestern Europe: Trends 2005-30. *Energy Policy* 35(10), 5206–5219.
- Hodges, S. D. (2004). The value of a storage facility. working paper. Warwick Business School.

- Holland, A. (2007). Optimization of injection/withdrawal schedules for natural gas storage facilities. In *Proceedings of the Twenty-seventh SGAI International Conference on Artificial Intelligence (AI-2007)*, Cambridge, England. Springer.
- Hull, J. C. (2009). *Options, futures, and other derivatives*. Upper Saddle River, New Jersey: Prentice Hall.
- IEA (2009). *International Energy Agency. World Energy Outlook 2009* (2009 ed. ed.). Paris.
- Kjärstad, J. and F. Johnsson (2007). Prospects of the European gas market. *Energy Policy* 35(2), 869–888.
- Longstaff, F. and E. Schwartz (2001, Sep). Valuing american options by simulation: A simple least-squares approach. *Review Of Financial Studies* 14(1), 113–147.
- Ludkovski, M. and R. Carmona (2010). Valuation of energy storage: An optimal switching approach. *Quantitative Finance* 10(4), 359–374.
- Manoliu, M. (2004). Storage options valuation using multilevel trees and calendar spreads. *International Journal of Theoretical and Applied Finance* 7(4), 425–464.
- Maragos, S. (2002). Valuation of the operational flexibility of natural gas storage reservoirs. In E. I. Ronn (Ed.), *Real options and energy management*, pp. 431–456. London: Risk Books.
- Mardia, K., J. Kent, and J. Bibby (1979). *Multivariate analysis*. Academic Press.
- Munz, C. and T. Westermann (2006). *Numerische Behandlung gewöhnlicher und partieller Differenzialgleichungen*.
- Schlüter, S. and M. Davison (2010). Pricing an European Gas Storage Facility using a Continuous-Time Spot Price Model with GARCH Diffusion. Working Paper.
- Schwartz, E. (1997). The stochastic behavior of commodity prices: Implications for valuation and hedging. *Journal of Finance* 52, 923–974.
- Secomandi, N. (2010). Optimal Commodity Trading with a Capacitated Storage Asset. *Management Science* 56(3), 449.
- Thompson, M., M. Davison, and H. Rasmussen (2009). Natural Gas Storage Valuation and Optimization A Real Options Application. *Naval research logistics* 56(3), 226–238.
- Weber, C. (2005). *Uncertainty in the electric power industry: methods and models for decision support*. New York: Springer.
- Weron, R. (2006). *Modeling and Forecasting Electricity Loads and Prices: A Statistical Approach*.
- Weston, T. (2002). Applying stochastic dynamic programming to the valuation of gas storage and generation assets. In E. I. Ronn (Ed.), *Real options and energy management*, pp. 183–216. London: Risk Books.

# Appendix

## Proof of the stability bound

To guarantee stability of the explicit finite differencing approach, the coefficients of  $V_{i,j+1}^{k+1}$ ,  $V_{i,j}^{k+1}$  and  $V_{i,j-1}^{k+1}$  in equation (20) must be positive (cf. Ames (1992)). Consequently we may write the following conditions:

$$\frac{1}{2} \frac{\sigma^2}{\Delta S^2} S_j^2 + \frac{\kappa(\theta - \lambda\eta - \ln(S_j)) S_j}{2\Delta S} \geq 0, \quad (33)$$

$$\frac{1}{\Delta t} - \frac{\sigma^2}{\Delta S^2} S_j^2 - \lambda - \rho \geq 0, \quad (34)$$

$$\frac{1}{2} \frac{\sigma^2}{\Delta S^2} S_j^2 - \frac{\kappa(\theta - \lambda\eta - \ln(S_j)) S_j}{2\Delta S} \geq 0. \quad (35)$$

In a first step the upper bound for  $\Delta S$  is derived applying conditions (33) and (35). Condition (33) can be reformulated as follows:

$$\begin{aligned} & \frac{\sigma^2}{\Delta S} S_j^2 + \kappa(\theta - \lambda\eta - \ln(S_j)) S_j \geq 0 \\ \Leftrightarrow & \frac{\kappa(\ln(S_j) - (\theta - \lambda\eta))}{\sigma^2 S_j} \leq \frac{1}{\Delta S} \end{aligned} \quad (36)$$

Considering inequation (36) we distinguish the two cases  $S_j \in [S_{\min}, e^{\theta - \lambda\eta} [$  and  $S_j \in ]e^{\theta - \lambda\eta}, S_{\max}]$ . For the first case, the left side of the inequation (36) is negative. Thus, as we consider only positive price steps  $\Delta S$  we limit the following considerations to  $S_j \in ]e^{\theta - \lambda\eta}, S_{\max}]$ . To obtain the most restrictive bound for  $\Delta S$  in this interval we rewrite inequation (36):

$$\frac{\sigma^2 S_j}{\kappa(\ln(S_j) - (\theta - \lambda\eta))} \geq \Delta S \quad (37)$$

and apply a first order condition to equation (37):

$$\begin{aligned} & \frac{\partial \left( \frac{\sigma^2 S_j}{\kappa(\ln(S_j) - (\theta - \lambda\eta))} \right)}{\partial S_j} \stackrel{!}{=} 0 \\ \Leftrightarrow & \frac{\sigma^2 \kappa(\ln(S_j) - (\theta - \lambda\eta)) - \sigma^2 \kappa}{\kappa^2 (\ln(S_j) - (\theta - \lambda\eta))^2} \stackrel{!}{=} 0 \\ \Leftrightarrow & \ln(S_j) - \theta + \lambda\eta - 1 \stackrel{!}{=} 0 \\ \Rightarrow & S^* = e^{\theta - \lambda\eta + 1}. \end{aligned} \quad (38)$$

Moreover, we analyse the second derivative to proof that the stability boundary attains its global minimum in  $S^*$  on the interval  $]e^{\theta-\lambda\eta}, S_{\max}]$ :

$$\begin{aligned}
& \frac{\partial \left( \frac{\sigma^2 \kappa (\ln(S_j) - (\theta - \lambda \eta)) - \sigma^2 \kappa}{\kappa^2 (\ln(S_j) - (\theta - \lambda \eta))^2} \right)}{\partial S_j} \tag{39} \\
&= \frac{\frac{1}{S_j} \sigma^2 \kappa^3 (\ln(S_j) - (\theta - \lambda \eta))^2 - 2 \frac{1}{S_j} \sigma^2 \kappa^3 (\ln(S_j) - (\theta - \lambda \eta) - 1) (\ln(S_j) - (\theta - \lambda \eta))}{\kappa^4 (\ln(S_j) - (\theta - \lambda \eta))^4} \\
&= \frac{\frac{1}{S_j} \sigma^2 \kappa^3 (\ln(S_j) - (\theta - \lambda \eta)) ((\ln(S_j) - (\theta - \lambda \eta)) (1 - 2) + 2)}{\kappa^4 (\ln(S_j) - (\theta - \lambda \eta))^4} \\
&= \frac{\frac{1}{S_j} \sigma^2 (- (\ln(S_j) - (\theta - \lambda \eta)) + 2)}{\kappa (\ln(S_j) - (\theta - \lambda \eta))^3} \\
&\stackrel{S_j=S^*}{=} \frac{\frac{1}{e^{\theta-\lambda\eta+1}} \sigma^2 (- (\theta - \lambda \eta + 1 - \theta + \lambda \eta) + 2)}{\kappa (\theta - \lambda \eta + 1 - \theta + \lambda \eta)^3} \\
&= \frac{1}{\kappa e^{\theta-\lambda\eta+1}} \sigma^2 \\
&\stackrel{\sigma^2 > 0, \kappa > 0}{>} 0.
\end{aligned}$$

Accordingly equation (36) attains a global minimum in  $S^*$  on the interval  $]e^{\theta-\lambda\eta}, S_{\max}]$ .

For condition (35) follows analogous:

$$\begin{aligned}
& \frac{\sigma^2}{\Delta S} S_j^2 - \kappa (\theta - \lambda \eta - \ln(S_j)) S_j \geq 0 \\
& \Leftrightarrow \frac{\kappa (\theta - \lambda \eta - \ln(S_j))}{\sigma^2 S_j} \leq \frac{1}{\Delta S}. \tag{40}
\end{aligned}$$

The left side of this inequation turns negative for  $S_j \in ]e^{\theta-\lambda\eta}, S_{\max}]$ . Following the above arguments we limit the following considerations to  $S_j \in [S_{\min}, e^{\theta-\lambda\eta}[$ . To obtain the most restrictive bound for  $\Delta S$  in this interval we rewrite inequation (40) as

$$\frac{\sigma^2 S_j}{\kappa (\theta - \lambda \eta - \ln(S_j))} \geq \Delta S \tag{41}$$

and consider the first derivative of equation (41):

$$\begin{aligned}
& \frac{\partial \left( \frac{\sigma^2 S_j}{\kappa (\theta - \lambda \eta - \ln(S_j))} \right)}{\partial S_j} \tag{42} \\
&= \frac{\sigma^2 \kappa (\theta - \lambda \eta - \ln(S_j)) + \sigma^2 \kappa}{\kappa^2 (\theta - \lambda \eta - \ln(S_j))^2} \\
&\stackrel{S_j < e^{\theta-\lambda\eta}}{\geq} 0.
\end{aligned}$$

Accordingly, the left side of inequation (41) grows monotonic on the bounded interval  $[S_{\min}, e^{\theta-\lambda\eta}[$ . As this function is continuous in  $S_j$  on this interval the minimum is attained on the left boundary of the interval:

$$S^{**} = \frac{\sigma^2 S_{\min}}{\kappa(\theta - \lambda\eta - \ln(S_{\min}))} \quad (43)$$

Combining (37) and (41) results in:<sup>17</sup>

$$\Delta S \leq \frac{\sigma^2 S_j}{\kappa|\theta - \lambda\eta - \ln(S_j)|} \text{ with } S_j \neq e^{\theta-\lambda\eta}. \quad (45)$$

Thereby, the minimum of the right side results as:

$$\min\{S^{**}, S^*\}. \quad (46)$$

In a second step the lower bound is derived by reformulating condition (34):

$$\frac{1}{\Delta t \sigma^2 S_j^2} - \frac{(\lambda + \rho) \Delta t}{\Delta t \sigma^2 S_j^2} \geq \frac{1}{\Delta S^2}$$

Assuming  $(\lambda + \rho) \Delta t < 1$  this can be reformulated:

$$\sqrt{\frac{\Delta t \sigma^2 S_j^2}{1 - (\lambda + \rho) \Delta t}} \leq \Delta S. \quad (47)$$

Combining (45) and (47) directly results in stability condition (21).

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<sup>17</sup>Excluding the price grid point  $S_j = e^{\theta-\lambda\eta}$  does not alter the minimum as:

$$\lim_{S_j \rightarrow e^{\theta-\lambda\eta}} \frac{\sigma^2 S_j}{\kappa|\theta - \lambda\eta - \ln(S_j)|} = \infty. \quad (44)$$